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THE RELATION OF SPACE AND GEOMETRY TO EXPERIENCE

VII. CONFLICTING MEASUREMENTS

IN OUR last lecture we defined vectors—i. e., directed distances—in terms of such notions as we had already taken as primitive in the definition of parallelism, and of no other notions. We then defined in terms of these notions and that of some particular spatial region alone, which we suppose to be spherical, in all future applications of this definition, the set of all vectors-in-the-sphere: that is, the class of all directed distances inside our sphere. We defined, again introducing no new notions whatever into our definition, the magnitude of each vector inside a given sphere in terms of the diameter of the sphere as a unit, or as we put it, the *index* of a vector-in-the-sphere. This definition of the index of a vector was so framed as to give perfectly unambiguous indices of the vectors in any region whatever, no matter how it might be shaped, in terms of that region, without involving, for example, that this region should have anything at all analogous with a diameter, but it was also so framed that, in an ordinary Euclidean space, if the "sphere of measurement" is really spherical in shape, the system of measurement of vectors-in-itself defined by it will agree completely with the system of measurement that is characteristic of ordinary Euclidean space.

Our purpose in this lecture is to obtain from the theory of measurement developed in the last lecture a general system of measurement by which we can determine the distance between any two points in space. In our last lecture, we left this problem still unsolved in two distinct ways. In the first place, the theory of measurement which we developed in the last lecture only told us how to measure the distance between two points both of which lie within our sphere of measurement. This leaves open the problem how we are to measure distances between two points of which one or both lie outside our standard sphere. In the second place, we have not discussed in any manner whatever the problem, how we are to determine what our standard sphere of measurement is to be, and how it is to be discriminated from other sets of generalized points. These two questions and the further problems to which they give rise form the subject-matter of this and a large part of the following lecture.

As we have just said, our first task is to find a method of extending the system of measurement which we have developed for all distances inside a given sphere to all distances in space, so that we may be able to compare any distance in space with the diameter of our standard sphere. You will remember that we defined a certain vector throughout space as the extension of a certain vector-in-a-sphere R, when and only when in all the cases where two points that are separated by R they are separated by the vector throughout space in question. It is clear that it is only natural to assign to a given vector throughout space that is the extension of R the index of R as an expression of the magnitude of the vector throughout space. Of course, we have said nothing which makes it a consequence of the definition of the extension of a vector-in-a-region that a vector in space cannot be the extension of vectors-in-a-given-sphere having distinct indices, though this cannot happen if our fundamental notions live up to their names. When this happens, we shall say that the vector in space in question possesses *both* of these indices, at least for the present. Since by this attribution of an index to the extension of a vector in our standard sphere we have found a way of measuring distances between points in any part of space, it might seem that we have obtained a satisfactory definition of a distance, which will apply to any distances whatever. A little reflection, however, will convince one that this is not the case. There are certain vectors which are not the extensions of any vectors in our standard sphere, in general. If our standard sphere is one inch in radius, it is obvious that no distance of two inches can be the extension of any vector inside our standard sphere, for there can be no two points both inside our standard sphere separated by a vector two inches long. The question is therefore before us, how are we to measure in terms of the diameter of our standard sphere distances larger than this diameter? This problem is a particular case of the more general one as to how we measure any magnitude with a scale smaller than the magnitude itself. We have that problem on hand, for example, when we wish to measure a yard and a half of cloth with a footrule. We solve this problem, as a matter of practice, as follows: we first apply the footrule to one end of the piece of cloth, and make a step of one foot. We then start from the end of the piece already measured and make another step of one foot. We find that after making in this manner four distinct successive steps, each a foot in length, provided that all these steps have been taken in a straight line pointing directly towards the further end of the piece of cloth, if we now take a step of only half a foot in length, we shall precisely reach the end of the strip of cloth. That is, if we take four successive steps of the whole length of the footrule, and so dispose them that we are led by them as near as possible to the end of the strip of cloth, we shall have to take a further step of the length of half a foot, measured by the rule, to reach the end of the strip of cloth. It is not essential, however, that our first four steps should each be a foot in length: we might first have taken a step seven inches in length, measured by the footrule, then one eight inches in length, then one eleven inches in length, then one ten inches in length, then one six inches in length, then one five inches in length, and finally, another step seven inches in length. That is, if we cover the space from one end of the strip of cloth to the other in a finite number of steps which are measurable by our footrule and which are so disposed that the sum of the lengths of these measurable steps is as small as possible, we call this sum of all the lengths of these steps the total length of the strip of cloth.

Let us now return to the problem of the measurement of distances which are too large to fit into our sphere of measurement. Suppose that the distance between the point A and the point B is of this kind. Then it will be of course impossible to go from A to B by a single step which belongs to the extension of some vector situated inside our standard sphere, but it may be possible to make the transition from A to B by the intervention of a finite sequence of successive steps each of which is an instance of the extension of some vector in our standard sphere. Let these steps be instances of the vectors which form the extensions of the vectors-in-our-sphere $S, S', S'', \ldots\ldots, S^{(n)}$, and let the index of $S^{(k)}$ be $i^{(k)}$, where $S^{(k)}$ stands for the *k*th step in the chain connecting A and B. Let the sum $i+i'+i''+\ldots\ldots+i^{(n)}$ be called I. I represents, then the total length of the chain of linear segments representing the steps by which the transition from A to B is made. The length I evidently depends on the particular chain which we select to connect A and B, and on what particular vector-in-our-sphere we regard as furnishing each member of our chain as its extension, when a member of our chain represents the extension of two distinct vectors-in-our-sphere. In ordinary geometry, there is a certain minimum length which chains connecting A and B actually possess—that is, I has a certain minimum value, and there are actually chains of a finite number of vectors which are extensions of vectors in our standard sphere which have this value of I as the sum of the indices of their members. Such chains stretch in a straight line from A to B: they exist, for every distance between two points which is in magnitude less than the diameter of our standard sphere may easily be shown to be an instance of the extension of some vector-in-our-sphere, and Archimedes' axiom holds in ordinary geometry—that is, since when any two distances *l* and *m* be given, there is some integer *k* such that the distance *kl* is greater than *m*, so that we may get anywhere by a finite number of steps as small as we please. If, however, the collection of generalized points which we select as spherical should turn out, after all, not to be spherical, or if our initial relation of apparent intersection among convex solids should belie its name, we have no proof at hand that there exists a chain connecting A and B whose length is actually the shortest that such a chain can have: there may be chains, for example, in which I may be made to assume any value you please greater than two, while there may be no chain for which I assumes precisely the value "two." It would seem highly unnatural to say that in this situation, which, as we have seen, can never occur in ordinary geometry, A is at no distance from B. We wish, therefore, to obtain a definition of the distance between A and B which will be the least possible value of I when such a value exists, and which will be, to put it crudely, sufficiently like that number to be called naturally the distance from A to B when there is no single value of

I which is smaller than all its other values. The quantity which we thus define as the distance between A and B must further be such as always to exist when A and B are ordinary generalized points. Furthermore, it must not be such as to make our system of measurement trivial too often, by causing too many distances equal to zero, or in some similar manner.

In our last lecture, we took notice of the fact that if we are given any set S of positive real numbers, and if all the members of S are less than some given positive number n, there is some single positive real number which we may call x, which is at least as great as any member of S, but which is also such that if e be any positive real number, however small it may be, there is some member of S greater than $n—e$. x is called the upper limit or maximum of S. Whether S be made up of numbers all of which are smaller than some fixed real positive number or not, it may be shown in a similar manner that S also determines a single positive real number v—which may be zero—which is smaller than any member of S, but which is such that if e be a positive real number as small as you please, $y+e$ is greater than some member of S. This number y, which is uniquely determined by the class S is called the *lower limit or minimum of* S. The existence of the lower limit or minimum of any class of positive or zero real numbers may be proved as follows: let S be the class in question, and let S′ be the class of all positive zero real numbers smaller than any member of S. S or S′ may or may not contain any terms. Let us suppose that S is not the null-class, the class with no members, and that it actually contains some terms. In this case, S′ may or may not contain some members. If S′ contains no member, it is obvious that e, however small it may be, is greater than some member of S: that is, $0+e$ is greater than some member of S. This is true because, by hypothesis, there is no positive or zero number smaller than every member of S, as such a number would belong to S′, which we suppose without members. Since S is made up entirely of positive or zero real numbers, it can contain no member less than zero. Consequently 0 is the lower limit or minimum of S. If S′ contains members, let y be the upper limit or maximum of S′. Then y is at least as great as any member of S′, and it follows, since we can easily show that S′ contains members which approach as closely as we choose to the members of S, that there are members of S smaller than $y+e$, however small a positive real number e may be. On the other hand, there can be no member of S smaller than y. For suppose that z is such a member of S. Then, in accordance with the way in which we have determined S′, since it is made up of all positive or zero real numbers less than every member of S, there can be no member of S′ greater than or equal to z: that is, there can be no member of S′ greater than or equal to $y—e$, where e is the positive number $y—z$. Consequently y fails to satisfy the definition of the upper limit or maximum of S′. But y is by definition the upper limit or maximum of S′, so that our supposition that there is a member of S smaller than y engenders a contradiction. We see as a result of this that y satisfies both of the conditions which go to make up the complete definition of the minimum or lower limit of S, so that there exists a minimum or lower limit of S in this case, as well as in that where S′ has no members. If we now consider the remaining alternative concerning the natures of S and S′—that is, if S contains no members whatever, as is the case when it is made up of all odd multiples of ten or of all integers that are commensurable with π—we shall, to simplify matters, make an *ad hoc* definition of the lower limit or minimum of S, and shall say that this lower limit or minimum is zero. This latter definition is, it is true, somewhat artificial. If we make this convention, it follows from what we have said that any conceivable collection of numbers has at least one minimum or lower limit: the uniqueness of this limit, to which we have already referred, may be demonstrated by a very simple proof, quite analogous to that which we gave last time for the uniqueness of the *maximum or upper limit* of a class of positive numbers.

We have just seen how it is always possible to assign to a given set of positive or zero real numbers one and only one positive or zero real number which is smaller than or at least as small as any member of the set, but to which the members of the set approach as near as we please, and we have called this number the minimum or lower limit of the set. Let us see whether we can define the distance between two points as the minimum or lower limit of the set of values of I for different values of I for the various paths connecting the points in question. In the first place, we must show that, provided that there is a smallest value of I, this must be the minimum or lower limit of all the possible values of I for the two points, since this condition is necessary if we are to regard the distance between two points as the length of the shortest path between them, as we do in ordinary space. This is true because the smallest of a set of numbers is at least as small as any member of the set, while there is no degree of approximation with which you cannot make it represent some member of the set, since it is itself a member of the set; consequently the smallest of a set of numbers, provided there is such a number, is the minimum or lower limit of the set. It results from this that the length of the shortest path from A to B —i. e., the least value of I for the two points in question—which we should naturally call the true distance between A and B, if it exists, is precisely that lower limit or minimum of the possible values of I which we have just agreed to call the distance between A and B. However, if there is no shortest distance between A and B along any path made up of the vectors that we have already measured—that is, if there is no least value of I—there must always, by what we have just seen, be a minimum or lower limit of the values of I, which will have many of the properties that are characteristic of a minimum value of I, and which will satisfy the definition which we have just formulated of the distance between A and B.

Given our sphere of comparison, then, we have thus been able to define in terms only of those fundamental notions which we have already explicitly formulated the distance between any two of those generalized points which correspond to ordinary geometrical points. Upon a slight investigation, we should find that we can prove, independently of any assumptions concerning the formal properties of the objects exemplifying our fundamental notions, a few simple geometrical theorems concerning distances. We can prove, for instance, that if A, B, and C are any three points, the sum of the distance from A to B and that from B to C is not less than that from A to C. We can also prove that the distance from A to B equals that from B to A. We cannot, however, prove from our definitions alone, that the distances that we have so far defined have *all* the formal properties of distances in ordinary geometry. In ordinary geometry, for example, if the mutual distances of four distinct points are given, and the distances of a fifth point from three of these are known, the distance of this fifth point from the remaining vertex of the tetrahedron formed by the four original points is determined to have one of only two possible values, while our definitions do not secure to any distance any precise quantitative relation to other distances, in general.

We have given the definition of distance just developed with the original intention of using it only for those distances that do not fit into our sphere of reference—the diameter of which, by the way, is still the unit of our sphere of measurement. If, however, we examine into the definition, we see that it may also naturally be applied to distances which fit into our sphere of measurement—for any two generalized points which are separated by a vector lying in our sphere of measurement, and consequently by the extension of this vector, are thereby connected by a chain consisting of that single step, and consequently have a distance in the sense in which a distance is defined in terms of a chain of vectors. It is further obvious that in a system of generalized points which behave in a decent geometrical manner, a chain connecting A and B and consisting of a single vector is at least as short as any other chain connecting A and B, so that the magnitude of the vector from A to B is the same as the distance from A to B, in the sense of the minimum length of a chain from A to B. We can consequently throw away our first definition of the distance from A to B entirely, and be sure that if we define all distances in space by chains, our definition will be natural—at least as natural, that is, as our direct definition of distances by means of vectors. We have thus attained a definition of all the distances in space which involves besides the notions that we have taken as primitive only that of a certain standard sphere of measurement.

The question now arises, what sphere is the actual unit sphere of our measurements? If it turns out that we can select no set of generalized points as the unique unit sphere, how are we to determine the various spheres which shall serve as the bases of our measurements, and if these spheres give us different values as expressions of the magnitude of a given distance, how can we reduce the various results which we obtain to a single internally coherent system of measurement? It is easy to indicate two directions in which we may approach this problem, corresponding respectively to the two definitions which we gave of parallel lines in a preceding lecture. It will be remembered that in the process of obtaining one of these definitions of parallel lines we defined a certain collection of sets of generalized points called a-spheres entirely in terms of our experience of the intersection of convex solids. We saw that if our experience of the intersection of convex solids records two convex solids as intersecting when and only when they both are situated in a certain region of space and approach one another closer than a certain distance which is the same everywhere in space, our a-spheres, *qua* sets of generalized points, will be spheres of a certain uniform size. This result will seem more familiar to you if I put it in another form. Many of you have seen the various processes of the differential and integral treated in the text-books of the more old-fashioned sort as if they dealt with operations of division or summation among certain entities called infinitesimals. These infinitesimals were regarded as magnitudes, all very small, but equal, in general. By the English mathematicians of the eighteenth century, who, like their philosophical colleagues, were of a more empiricist attitude to their subject than those of the Continent, these infinitesimals with which the calculus seems to deal were regarded as if they were the smallest objects accessible to our direct sensory experience. Since it was an essential property of the infinitesimals, in their mathematical use, that they should be equal, in general, those mathematicians who held this view were driven to the supposition that all just-noticeable sensible objects—all *minima sensibilia*—are equal. Now, it will be remembered that our a-spheres bear a very close analogy to the *minima sensibilia* of the philosophers, in that they mark the lower boundary of the sensibilia we are considering—namely, convex solids —with respect to their magnitude. Our assumption that all a-spheres are equal is consequently one of those things which was involved as an unquestioned presupposition in the empiricist treatment of the calculus, although it involves the admission of far less than is necessary for the support of that view. The fact that it is such a natural hypothesis, and that it is an element in a view of the greatest historical importance, makes the consideration of a theory of measurement based on this hypothesis a thing of the utmost interest, whether it be strictly true or not that all *minima sensibilia* are what we should ordinarily regard as of equal magnitude. Let us first notice, however, that it is extremely improbable that the delicacy of our sensory discrimination, and as a corollary the size of our *minima sensibilia,* is in any ordinary sense the same throughout all those parts of space which are more directly accessible to our sense-experience.

On the assumption that we could naturally call all a-spheres equal, and that our space has all the normal geometrical properties, it is a matter of indifference which a-sphere we use as a unit of measurement, and all the diameters of all a-spheres are equal. It will not, however, be in general a matter of indifference which a-sphere we take as our standard sphere, if this assumption is not fulfilled. In such a case, it may be possible to get several numbers, all of which express with equal right the distance between two points, A and B, according to the a-sphere which we choose as our unit of measurement. If we are to be able to regard some single number as the only true distance AB, as measured in terms of the diameter common to all a-spheres as a unit, we must find some way of obtaining from all these different measures of this distance some single quantity, which is uniquely determined by all these various measures, and which coincides with their common value if they agree. There is, of course, a certain degree of arbitrariness in our selection of this distance, but one may easily show that if we regard the true distance AB as the minimum or lower limit of all the values which we can get for the distance AB by using various a-spheres as standard spheres, we shall obtain an entirely unequivocal definition of the distance AB. In the first place, this distance will always exist, be finite, and uniquely determined, since, as we have seen, all these things are true of the minimum or lower limit of any set of positive or zero numbers. In the second place, if all the distances of A from B, measured by a-spheres, agree, they will all coincide with their minimum or lower limit, as one may see on inspection. We have thus obtained a definition of the distance between any two points in space which involves no notion other than that of our experience of the intersection of convex solids, which secures that any two points in space shall be at one and only one distance from one another, and which will completely agree with our usual notions of the distance between two points, provided that our experience of the intersection of convex solids has such properties as we should naturally expect it to have, and provided, moreover, that we can naturally call all *minima sensibilia* equal.

In the system of measurement thus obtained, some of the theorems of ordinary geometry will still hold good, irrespective of any assumptions about the nature of our fundamental experience; for example: the distance AB will equal the distance BA, the sum of the distances AB and BC will be at least as large as the distance AC, and so on indefinitely. For the most part, however, the geometrical properties of our distances will be dependent on the nature of our experience of the intersection of convex solids, and will not, in general, agree with those characteristics of ordinary Euclidean geometry, if our fundamental experience has not such properties as we should naturally expect it to have. For example, the theorem that, if any tetrahedron is given, the distance of a point from one of its corners must assume one of a certain pair of values, once the distance of the point from each of the other three corners is known, which is true in ordinary geometry, cannot be deduced from our definition of distances alone.

However, the object of this course of lectures is to show how we can eliminate from geometry all presuppositions concerning the particular formal properties of our original notions, and yet define our points, lines, distances, etc., in terms of these in such a manner that all the ordinary formal geometrical properties of these latter entities should follow from their definitions alone. To do this, we must be able to obtain a definition of distance which will depend on our experience of the intersection of convex solids alone, and which will of itself be a sufficient guarantee that our distances satisfy the usual geometrical laws, and it may further be shown that, once we have a perfectly satisfactory definition of all the distances in space, we have it in our power to give perfectly satisfactory definitions of all geometrical entities. As we have just seen, we have not yet succeeded in completely performing this task; the problem, however, is an extremely interesting one, and one of the utmost philosophical importance. Let us turn our attention to it for a little while.

Our last definition of distance has practically indicated to us how we should make a survey of the universe, for it correlates with every pair of ordinary generalized points the distance between them. However, it will be a bad survey, in general. By this I mean distances which it causes to separate points will not check up, will not "gee" with one another. You all know that a surveyor always makes more observations than are merely sufficient to indicate the position of each point he is mapping a single time only: he observes the same point five or six times over from various observation-stations, and records all his observations. In an ideally perfect survey, all these observations would come out in complete accord with one another, but as a matter of fact, they seldom or never come out in complete accord with one another in any actual survey. The several determinations of each triangulation-station indicate on the map, not one point, but a number of points, situated in more or less close proximity to one another. It is a part of the task of the surveyor to fix a single point, which may be said to be the best representative of these several points, in terms of the collection made up of all of them. This he does by means of a certain mathematical theory known as the theory of least squares. This theory does not concern us here, except in so far as it fulfills the function of reducing a set of unharmonious measurements to a harmonious system; we shall have nothing to say concerning its technical details. Its function is to make every single point which forms a station in a certain survey correspond to one point on the map representing the results of this survey, and to one only. Then the surveyor can go and take the distances between these uniquely determined points on his map, and say that they represent—when the appropriate alterations of scale are made—the distances between the points on the earth's surface corresponding to the points on the map, and we may be *a priori* sure that the proper geometrical relations will hold among the distances so determined.

In order to make the discussion of surveying which we have just given more simple to grasp, we have been guilty of a slight inaccuracy of statement. We said that the several determinations of each triangulation station indicate on the map, not one point, but a number of points. As a matter of fact, however, the several determinations of each triangulation-station indicate no specific point whatever on the map until we have already determined a certain correspondence between points on the map and points on the surface of the earth. This presupposes that we are able to locate certain points on the map before others and to discriminate between such as we shall locate first and those which we locate later. We do not possess, however, any criterion for such a location. It is consequently necessary for us to possess some means of deriving at one blow from our confused measurements of the distances between our stations quantities which we can regard as the expression of the magnitudes of these distances, among which the proper geometrical relations will of necessity hold, if we are to be able to map our triangulation-stations in such a manner that to each triangulation station there will correspond one and only one point on the map. This task can also be accomplished with the aid of the method of least squares, in the case of any survey involving only *a finite number of points.* From any survey, then, however bad it may be, provided that it involves only a finite number of points, a set of finite quantities may be obtained, one of which is correlated with each pair of the points surveyed, and among these quantities the formulae which correlate distances in an ordinary Euclidean space will hold.

However, the bad survey of the universe which was made by our definition of distances in terms of a-spheres involves the measurement, not of a finite number of distances, but of an infinite number of distances, provided that the space that we have defined in terms of our experience of the intersection of convex solids is comparable in richness with the space of ordinary geometry. Now, the method of least squares is unable, as far as I know, to bring order into surveys that are bad at an infinite number of points. We do not at present possess a method of deriving a well-behaved Euclidean set of distances from the hodge-podge set of distances which we obtain, in general, from our definiton of distances in terms of a-spheres. I see no reason, however, for supposing that the problem of obtaining a method which should perform the same function for infinitely irregular systems that the theory of least squares fills for finitely irregular systems should be essentially insoluble. The solution of this problem is of absolutely vital importance for the philosophy of space, for however we may define distance, nothing is more certain than that the distances of points in space, as we first learn and observe them, only approximately satisfy the laws of geometry, yet we are absolutely sure that there are actual distances which our observed distances represent and which satisfy these laws precisely. If we were enabled by a theory analogous to that of least squares, to derive from our chaotic observed distances, distances whose geometrical properties and relations should be secured *a priori,* we could immediately explain this phenomenon, on which so much importance has been laid by philosophers.

In this lecture, we have extended the system of measurement determined by a sphere from its inferior to the whole of space. On the basis of that definition of parallelism which starts with a-spheres, we saw that a system of measurement could be developed, but that this would have certain defects. We saw that this system of measurement does not secure *a priori* the geometrical properties of space, but that it would only require the formation of a branch of mathematics which should deal with problems essentially similar in character to those dealt with in an already existing branch to give us a space whose geometrical properties would be *a priori.* In our next lecture we shall take up the problem of defining distances on the basis of our alternative definition of parallel lines, and we shall finally discuss the philosophic problems to which both these systems of measurement give rise.

VIII. Geometry an a Priori Science

IN OUR last lecture, we developed a theory of measurement upon the basis of that definition of parallel lines which involved the notion of a-spheres. In this lecture, we shall first develop an alternative theory of measurement, in which we follow out that definiton of parallelism which involved, not only the relation of apparent intersection among convex solids, but also four selected convex solids, which we agreed to call "unit spheres." If the distances of any point in space from the centers of these four spheres are known, and our four spheres, to put it in ordinary geometrical terms, are four spheres whose centers are not all coplanar, the position in space of the point in question is completely determined, and if the positions of two points are determined in this manner, their distance from one another is completely determined. As we intend in what follows to secure the proper geometrical interrelations of our distances, and to cause the theorems of geometry to be satisfied, by means of defining the position of any point in space in terms of its distances from the four centers of our standard spheres, the first notion which we must define is that of the distance of a point from the center of a sphere. Now, it is obvious that, whether a point x is inside a sphere S or not, the furthest point in S from x lies on the line through x passing through the center of S, and is that one of the points on the surface of S and also on the line in question which is further from x. If y be the point in S furthest from x, it therefore follows that the distance from x to y is the distance from x to the center of S, plus the radius of S. One might consequently think that we might define the distance of x from the center of S as the distance of x from the most remote point of S, minus the radius of the sphere. We wish, however, to have our definition applicable even where S is not precisely a sphere, and we consequently do not wish to make the assumption that there is any single point of S most remote from x an essential condition of the usefulness of our definition. Now, if there is any single point in S remote from x, the distance of this point from x will be the maximum or upper limit of the distances of points in S from x. However, it will be much more usual, in general, for there to exist a maximum or upper limit of the distances of points in S from x than for there to exist some single point at precisely that distance from x and belonging to S. We shall consequently regard the distance of a point from the center of S as the maximum or upper limit of the distances of points in S from the point in question, minus the radius of S. Now, if we make S itself our standard sphere of measurement, in the sense explained in our last two lectures, since the diameter of S will be our unit, the radius of S will be one-half. Consequently, it will be natural to regard the distance of a point from the center of S as the maximum or upper limit of the distances of points in S from the point in question, minus one-half. This definition, as may be readily seen, involves no other notions than such as we have already taken explicitly as primitive, together with such notions as "one-half," or "limit," which, as the modern logicians have shown, are purely logical notions.

Given the four equal, non-coplanar spheres that served as the foundation of our second theory of parallelism, we are now able to define the distances of any point we please from the center of each of them, in terms only of such notions as we have already taken explicitly as primitive. It is interesting to note that our definition involves absolutely no reference to such entities as the centers of these spheres. Be that as it may, we can treat our definitions as if they yielded us the distances of a point in space from four given points in space—the centers of our four spheres. We intend to use these four distances as the coördinates of our points, and to determine the position of any point in space by means of these four distances. We finally desire to determine the distances of any two points in space from one another in terms of the distances of each of the points from each of the four centers of our equal spheres. Our actual coördinates will not, however, be the four distances of a point from our four centers, since we wish our coördinates to be absolutely independent of one another, as this will facilitate such a definition of the distances in space as will secure automatically their satisfaction of the laws that bind distances to one another in ordinary geometry. Now, in ordinary Euclidean space, if we know the distance of a point from each of the vertices of a triangle, we know that its distance from any fourth point in space can assume one of only two possible values. We shall consequently determine a point, not by its distance from each of our four centers, but by its distance from three of our centers, and by whether its distance from the fourth center has its greatest possible value, the other three distances being given, or not. We shall, for example, write the coördinates of a point whose distance from the center of the standard sphere A is a, whose distance from the center of the standard sphere B is b, whose distance from the center of the standard sphere C is c, and whose distance from the center of the standard sphere D has its maximum value, a, b, and c being fixed, in the form $(a, b, c, +)$, while if the distance of our point from the center of D has not its maximum value, and everything else remained unchanged, we should represent our point by $(a, b, c, -)$. In these definitions, we have introduced no notion that we have not already taken explicitly as primitive.

The three coördinates and a sign by which we determine a point are completely independent of one another if our space already obeys the laws of ordinary geometry. If, however, we know the coördinates of two points, we do not yet know, by these data alone, the distance by which the two points are separated. This knowledge depends further upon a knowledge of the shape and size of the tetrahedron formed by the four centers of our spheres of reference—that is, on a knowledge of the remoteness of the four centers from one another. To discover this, we must already possess a definition of the distance between the centers of two unit spheres. A definition which is in every way analogous to the definition which we have already given of the distance of a point from the center of a sphere reads as follows: the distance between the centers of two spheres, S and T, is the maximum or upper limit of the distances between points in S and points in T, whether measured with reference to S or to T as the standard sphere, minus the radius of S plus the radius of T, which is one, since the diameters of S and of T are regarded as possessing a common value, and this common value is taken as our unit of distance.

We now possess sufficient data to transform the system of coördinates which we have already obtained into an ordinary Cartesian system of coördinates. If we form certain functions of the coördinates already defined for a point *p* in a perfectly determinate manner—which, it is true, are a little too intricate for us to exhibit here—and call these X, Y, and Z, we may be sure that, if our points, lines, distances, etc., have such properties as we should naturally associate with the names we give them; and if our four "spheres" are actually four equal spheres whose centers do not all lie in the same plane, our coördinates X, Y, and Z, will be an ordinary set of rectangular Cartesian coördinates whose axes are determined in a certain definite manner by our four spheres of reference, and whose unit of distance is the common diameter of all four standard spheres. We may also be sure that even if our space is not so completely subject to the ordinary laws of geometry, many of our points will, in general, still determine coördinates, and any point that determines the coördinates X, Y, and Z will determine them uniquely. We shall call the values of X, Y, and Z that are determined by a certain point *p* the *fundamental coördinates* of *p*.

It is by no means a necessary consequence of the definition of the fundamental coördinates of a generalized point that to a given point there must always correspond some set of coördinates. Let us call those generalized points that have fundamental coördinates *proper* generalized points. By the definition of a proper generalized point, it must have one set of fundamental coördinates and one only. It does not follow from this, however, or from anything else we have yet said, that no two distinct proper generalized points have the same set of fundamental coördinates, whatever the formal properties of our fundamental notions may be. We desire, however, to obtain ultimately a definition of a spatial point which will secure by itself that no two points hold the same position in space—that is, that no two points have the same fundamental coördinates. How shall we do this? We shall do it by introducing a new definition of a point in place of our definition of a generalized point, just as we formerly introduced the definition of a generalized point in place of our very first definition of a point. Our final definition reads as follows: a *revised point,* we shall say, is a class other than the null class, or class without any members—of all the proper generalized points that have a given set of fundamental coördinates. Obviously, we assign this set of coördinates to the revised point in question. It is evident, then, that no two distinct revised points can have the same set of coördinates, for otherwise they would coincide. A revised point, then, must determine a set of coördinates different from that of any other revised point; it further follows from the mode of formation of a revised point that every revised point has a set of coördinates, and that no revised point can have more than one set of coördinates. The appropriateness of calling revised points points will become obvious if you reflect that in an ordinary, well-behaved space a revised point will contain only one member —a certain generalized point—and that the revised point will consequently represent the same point, in our everyday sense of the term (whatever that sense may be) as the single generalized point which is its member. We have thus obtained a set of entities which can naturally be called points, between which and certain sets of coördinates or triads of numbers there subsists a certain correlation which is one-one, and extends over all of our revised points. In ordinary geometry, the correlation between points and their Cartesian coördinates is not only one-one, but also connects all the points in space with all possible sets of coördinates. Consequently, if our revised points are to have all the formal properties of the points in ordinary geometry, we must show, over and above what has been already shown, that every set of Cartesian coördinates—every triad of real numbers—determines some point. Now, it does not follow from the definition of a revised point that we have just given that every set of Cartesian coördinates determines a point. The manifest and obvious way to remove this imperfection in our system of definition is to find some entities that will fill the gaps in our system of points which are left by the absence of a revised point corresponding to a given set of coördinates. We can do this in the following manner: if to a given set of coördinates there corresponds a revised point, we shall say that the revised point is the point, in our final sense of the word, that corresponds to it, but if there is no revised point corresponding to a given set of coördinates, we shall call the set of coördinates itself the point that fills the position indicated by our set of coördinates. If we thus interpolate sets of coördinates as points into our system, we shall find that our complete set of points will be in one-one correspondence with our complete set of coördinate triads. We shall thus have obtained a space which agrees perfectly in this respect with the space of ordinary geometry.

Now, it is familiar to all those of you that have had an elementary mathematical training that all the theorems of geometry my be reduced to purely algebraic theorems, entirely independent of space and dependent only on the properties of number and quantity, when once a system of Cartesian coördinates has been set up. It has been shown by the modern mathematical logicians that all theorems that deal with number and quantity alone are theorems of pure logic, and are independent of any concrete experience whatever. Since these things are the case, then once a system of Cartesian coördinates has been defined in terms of our fundamental notions, we have done all that is necessary to prove the *a priori* character of geometry, for we can now so define lines, planes, circles, angles, etc., in terms of our set of coördinates—and ultimately in terms of those notions that we have explicitly taken as primitive —that all the theorems of geometry shall result from these definitions and the laws of algebra alone. Consequently, if the space of our every-day life is constituted of the entities that I have just called revised points and of the number-triads of the space that I have just defined, the theorems of geometry are *a priori* true, even though space is actually a function of experience.

This brings me to the end of the technically logical and mathematical portion of the present course. Let us consider what we have accomplished, and let us see in how far we have fulfilled the purpose which this course was designed to fulfill, and have proved or rendered more probable the thesis which we set out to prove. The thesis of this course of lectures is that, whereas space is a function of experience, the geometrical properties of space are *a priori* certain, and may be proved without involving any reference to the concrete nature of the experience of which space is a function. We claimed that the geometrical properties of space were due to the method of schematization by which space is obtained as a function of experience. To show that such a situation is possible, one thing that must be done is to exhibit a system in which geometrical properties are the results of a method of schematization applied to an arbitrary subject-matter, and of this method of schematization alone. We can fairly claim to have accomplished this task by exhibiting the spatial system resulting from our second definition of parallelism, for in this we have introduced no notion pertaining to a certain spatial system—no notion, indeed, other than that of a certain relation, which we *call* the relation of experienced intersection among convex solids, and four arbitrary terms or sets of terms that enter into this relation, which we call four equal spheres whose centers do not all lie in a single plane—and notwithstanding the arbitrariness of the formal properties of our fundamental notions, we have so framed our definitions that certain of our entities must of necessity possess the formal properties characteristic of geometrical objects. We can further claim that we have pointed out that it would not require the solution of any problems generally different in character from those solved in already existing branches of mathematics (such as the theory of least squares), to derive a system of entities obeying all the formal properties of geometrical entities from certain relations or facts essentially similar to those more or less directly accessible to our experience, in such a manner that no presuppositions concerning the formal properties of these relations and facts are involved. That is, we have made it seem very probable that, even if we did already possess a geometry, and had no knowledge whatever of points, lines, planes, etc., we could build up from those facts ascertainable by means of a more or less direct experience, by a method involving only such notions as belong to logic and are not dependent on any concrete experience, a system of entities which one could naturally call lines, points, planes, etc., in such a manner that one could be *a priori* sure that the formal properties of these lines, points, planes, etc., would be those laid down by the laws of ordinary geometry. All this, though suggestive of the actual relations that subsist between geometry and experience, is not in any way conclusive evidence as to the nature of these relations. We have given no reason to justify us in supposing that the entities that we have called points, lines, planes, etc., are the same entities that we call by those names in our every-day life. Indeed, the feeling of artificiality that haunts us at every step of the ground we have covered seems to forbid this view, and I think that one will be perfectly justified if he categorically denies that the particular entities that we have called by various geometrical names are those that are called by those names in our every-day life. But the fact that there is nothing inherently impossible in the formatioin of a space whose geometrical properties shall be *a priori* certain, yet which will be a function of experience alone, and the absence of any other existing view of the relation between space and experience that will explain the association of certain geometrical entities with certain empirically known physical entities, and in addition thereto the non-experimental nature of geometry, entitles us to say that we have rendered the view that geometry deals with some schematization, some systematization, some arrangement of experience highly probable. We have given good reasons for supposing that the apriority of geometry is genuine, but that it is an apriority of method, not of subject-matter, and we have illustrated how an apriority of method which is not apriority of subject-matter is possible in the field of geometry.

We have the question still before us: if space, as we have said, is a schematization of experience, what sort of a schematization is it actually, and how is this schematization related to that which we have exhibited in this course of lectures? The first thing to notice is that it is very highly probable that the schematization by which the space of our every-day life is formed is probably not a fixed, immutable schematism, for which there is one analysis that is always right, while all the other accounts of its structure are always wrong. For instance, the schematization by which the space of a carpenter is reached is necessarily different from that by which a physicist attains his space: one might almost say that with the physicist, his straight lines are his light-rays, for a light-ray in a vacuum is the criterion by which he tests the straightness of anything else, and consequently it is at least reasonable to suppose that the complicated synthesis and organization of experience through which he must go in order to obtain a light-ray—since light-rays, as such, are not given in experience—must play a part in the construction of the lines of his geometry; while this obviously cannot be the case with the carpenter, whose criteria of straightness are simply chalk-lines and T-squares. It seems likely, moreover, that even with one and the same person, the schematism he uses may vary with the problem which he is attacking: on one occasion, the physicist may find it more convenient to regard a point as if it were built up by some process of organization similar to that by which he organizes his experiences under the form of a light-ray; on another occasion, his method of schematization may be more analogous to that by which he obtains a gravitational line of force. It may be incorrect to speak of "space" as something unique: all that we ought to mention may be "spaces."

On the other hand, I feel fairly sure that certain tasks will have to be performed by all the divers methods of schematization that may lead us from experience to space. For example, the problem to which we devoted our third and fourth lectures—the problem, namely, of proceeding beyond that portion of space which is more or less directly accessible to our experience, and of obtaining definitions of points and of lines which will yield us all the points and lines in space—is one that is inseparable from any method of schematization by which we obtain space from experience, and I believe that the method of making this extension of space that we there developed is essentially similar to that which we use in our actual processes of obtaining space. The problem of an infinite theory of least squares, of turning a bad survey of the universe into a good one, is one that we must meet in almost any process of deriving space from experience. The problem of distinguishing parallel lines from intersecting lines is another problem that is not confined to the system developed in this course. It will thus be true that though there are many different systems of schematization which, on certain occasions, yield us space as a function of experience, and while the method which we have described in the preceding chapters may be different from all of these, there will be certain problems that run through all these methods and the one which we have given, so that an exposition of the method which we have given cannot but throw light on the methods of schematization by which space is actually obtained.

It may seem to you that these tasks which I have just mentioned as essential steps in the attainment of space as a function of experience are, as our introspection shows us, not performed consciously, while they are too intricate to be performed unconsciously. I think, however, that both of these statements are open to question. In the first place, whenever a man thinks clearly enough to give a definite criterion for the straightness of a line, or the intersection of two lines, or whatever other geometrical property you please of physical, sensible objects, he is really simply rendering explicit some stage in the synthesis of experience which constitutes his space, and in so far as he has succeeded in forming his criteria of straightness, intersection, etc., into a system which must, by an internal necessity, be coherent, he has succeeded in making the complete method of schematization of his space determinate. We have already referred in this course of lectures to the fact that the surveyor, when he obtains a coherent map by a definite method from a mass of disharmonious data, is doing something of essentially the same nature as what we have been trying to do in this course of lectures, for he possesses a method by which he can deduce a map in which the laws of geometry hold from the most hopelessly incoherent and lawless set of observations or experiences. In the second place, it is a well-known fact that the complexity of a mental process is in itself no absolute bar to its unconsciousness. A man may have the most complicated set of criteria by which he determines the straightness of a given line or the flatness of a surface, but he may never have introspectively considered the nature of this process. It may turn out that many or all of the geometrical properties of his space result from these criteria alone, and that these criteria or definitions of straightness, flatness, etc., are so formulated that they constitute a perfectly coherent system, without his ever explicitly knowing that he uses these criteria or definitions at all. It does not seem to me at all unlikely that the mathematician or physicist, who unconsciously performs such intricate processes of reasoning as differentiation or integration, should perform unconsciously some comparatively simple synthesis whereby space, with all its geometrical properties, is obtained as a function of experience. As to the non-mathematician, who is unable to follow a complicated train of reasoning even consciously, and who can formulate only vague and unclear definitions, there is no reason to suppose that in the space which he obtains as a result of his own process of synthesis from experience the laws of geometry hold in any but a rough and vague and rough way. In short, it is perfectly possible for the methods of schematization by which space may be obtained from experience to be unconscious, at least in the sense that we never see it as a systematic whole.

In closing this course, I wish to make a few remarks about the manner in which the theory of space we have developed has answered that problem which we found the Kantian view of space unable to meet. I refer to the problem of the correspondence of certain physical objects with certain spatial entities. How can we use one of the geometrical lines that we might define in terms of our revised points, for instance, as a criterion of the straightness of such a convex solid as a mark on the blackboard with a piece of chalk may be? The answer to this question is very simple. We have already seen how we can regard a convex solid as a set of points, in our first sense, and we have also seen how we may make a set of points in our first sense determine uniquely a set of generalized points. We are thus able to regard a convex solid as if it were a certain set of generalized points. It is easy to carry this process a little further, and to regard a convex solid as a certain set of revised points determined uniquely by it, which may be regarded as the set of all the revised points inside the convex solid in question. We can then give a simple mathematical definition for the accuracy with which the convex solid in question represents a straight line: we can say, for example, that the linearity of a set of points is the ratio of the longest linear segment connecting two points of the region or set to the longest perpendicular segment that contains two points of the set. On Kant's view, it is essentially impossible that anything of the sort should be done, since his lines, forming portions of space, can be given to us only *a priori,* so that we cannot recognize them in an empirical situation, and associate them in a definite manner with empirical objects. We have thus completely established our case against Kant; our case against Mach and the whole Empiricist school of philosophers of mathematics has already been made out.

NORBERT WIENER.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.